MCWP Senior High Mathematics League

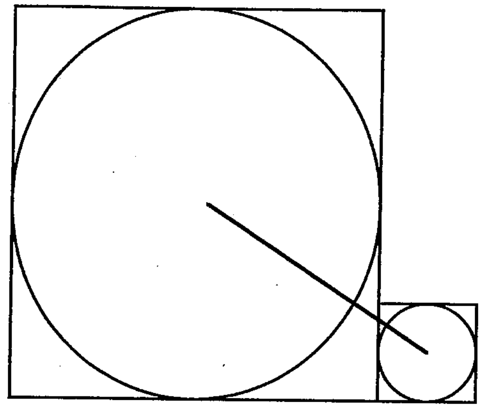
Contest #1 **October 16, 2006**

TEAM QUESTIONS

TIME LIMIT = 10 MINUTES

T-1. In a convex polygon of n sides, one interior angle contains $x°$, while each of the remaining $n-1$ interior angles contains $133°$. Compute all 4 possible values of x.

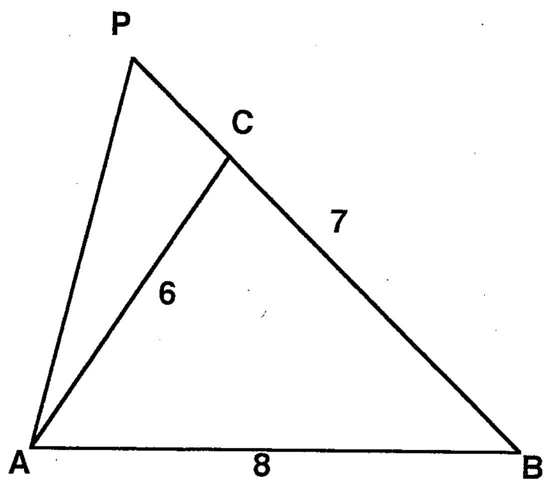
T-2. The area of the larger square is 49 times larger than the area of the smaller square. The total area of the two squares is 200 square units. Determine the length of the segment joining the centers of the two inscribed circles.



T-3. A dart is thrown at random at a circular board of radius 12 inches. Assuming that the dart always hits the board, what is the probability that the dart will be no closer than 2 inches and no farther than 6 inches from the center of the board?

T-4. In ΔABC, $AB = 8$, $BC = 7$, $CA = 6$, and side BC is extended to point P, as shown in the diagram, so that ΔPAB is similar to ΔPCA. Compute the length of segment $\overline{PC}$.

(NOTE: THE DIAGRAM IS NOT DRAWN TO SCALE.)



T-5. Three tennis balls snugly fit in a cylindrical can. The balls touch the side, top and bottom of the can. What is the ratio of the volume of the balls to the space in the can that is around the tennis balls?

MCWP Senior High Mathematics League

Contest 1 October 16, 2006

Individual Questions

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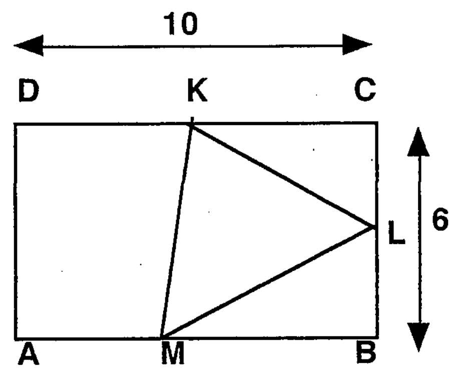
INDIVIDUAL ROUND 1 — 15 MINUTES

I–1. A line goes through the points $(1,3)$ and $(-2,k)$. For what value of k will the line also pass through the point $(7,-4)$?

I–2. A box contains four \$10 bills, six \$5 bills, and two \$1 bills. Two bills are taken at random from the box without replacement. What is the probability that both bills will be of the same denomination ?

I–3. In the rectangle $ABCD$, $AB = 10$, $BC = 6$, K is the midpoint of $\overline{CD}$, L is the midpoint of $\overline{BC}$, and M is on $\overline{AB}$ such that the area of ΔKLM is 12. Compute the length of segment $\overline{AM}$.



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INDIVIDUAL ROUND 2 — 15 MINUTES

I–4. The consecutive angles of a trapezoid form an arithmetic sequence. If the smallest angle is $75°$, find the number of degrees in the largest angle.

I–5. Suppose x and y are inversely proportional and positive. If x increases by $p\%$, then y decreases by what percent?

I–6. When the Speakers Club stopped accepting new members, it boasted a total of 500 members, 99% of which were male. Now, one year later, no new members had joined the club. However, some males, but no females, have withdrawn from the club, so that the club membership is now only 96% male. How many members are in the club now?

MCWP Senior High Mathematics League

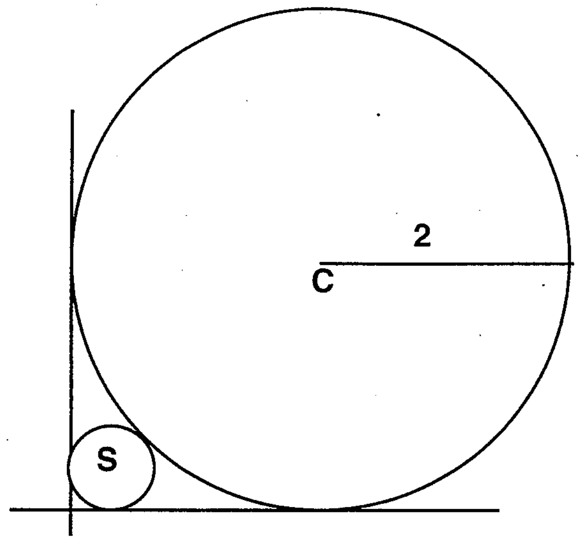
Contest #2 **November 13, 2006**

TEAM QUESTIONS

TIME LIMIT = 10 MINUTES

T-1 Mr. Magoo gave an exam in which the average mark of those passing was 65, and the average mark of those failing was 35. If the average mark of all participants was 53, what percentage of the participants passed? Assume that a 60 or higher is passing.

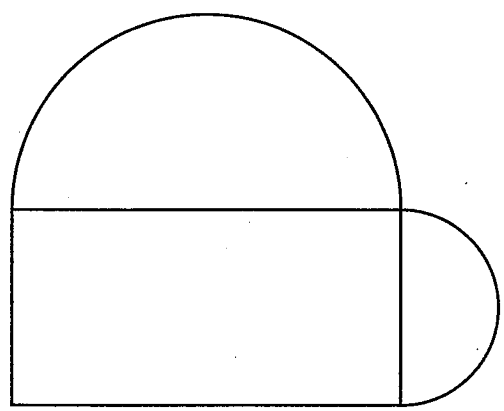
T-2. Let C be a circle of radius 2 that is tangent to two line segments that form a 90° angle. Another circle, call it S, is tangent to the same two line segments and to circle C. What is the radius of circle S?



T-3. What are both ordered pairs of integers (x, y) that satisfy

$$(1 + x + y)^2 = 1^2 + x^2 + y^2 \text{ ?}$$

T-4. In the figure shown at the right (not drawn to scale), the adjacent sides of a rectangle form diameters of two semicircular regions. The rectangular region has an area of 81. If the ratio of the areas of the semicircular regions is 16 to 1, what is the perimeter of the rectangle?



T-5. If a and b are positive integers, what is the smallest value of b for which $\dfrac{2005}{2006} < \dfrac{a}{b} < \dfrac{2006}{2007}$?

MCWP Senior High Mathematics League

Contest #2 **INDIVIDUAL QUESTIONS** **November 13, 2006**

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INDIVIDUAL ROUND 1 — 15 MINUTES

1. There are two values of x for which a triangle with side-lengths 10, 10, and x has an area of 48. One of these values is $x = 12$. What is the other value?

2. I have 3 boxes. One box contains 1 dime and 1 quarter, one contains 1 dime and 2 quarters, and one contains 2 dimes. If a coin chosen at random from a box chosen at random is a dime, what is the probability that the randomly chosen box contains at least 1 quarter?

3. A *cevian* (pronounced *chay-vee-in*) of a triangle is a segment that joins a vertex to any point on the opposite side except a vertex. If 1 cevian is drawn from each vertex of $\triangle T$, and if there's no point inside T where all 3 cevians intersect, then into how many disjoint, non-overlapping regions do the 3 cevians partition the interior of $\triangle T$?

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INDIVIDUAL ROUND 2 — 15 MINUTES

4. What are all values of x which satisfy $x + 3 - \frac{1}{x-1} > -x + 2 - \frac{1}{x-1}$?

5. In set $S = \{(1,5), (2,8), (3,7), (4,4), (5,6)\}$, every ordered pair has the form (x,f). What is the mean value of the variable x which occurs with frequency f?

6. Determine, in simplest form, the value of the binomial expansion

$$86^5 - 5(86)^4(87) + 10(86)^3(87)^2 - 10(86)^2(87)^3 + 5(86)(87)^4 - 87^5.$$

Contest #3

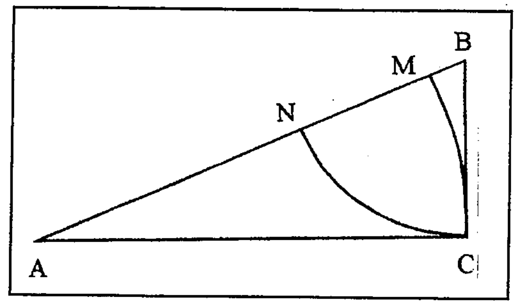
MCWP Senior High Mathematics League

December 11, 2006

TEAM QUESTIONS

TIME LIMIT = 10 MINUTES

T-1. In right triangle ABC with legs 5 and 12, arcs of circles are drawn as shown, one with center A and radius 12, and the other with center B and radius 5. Compute the length of segment $\overline{MN}$?



T-2. If $\dfrac{1}{a(b+1)} + \dfrac{1}{b(a+1)} = \dfrac{1}{(a+b)(b+1)}$, what is the value of $\dfrac{1}{a} + \dfrac{1}{b}$?

T-3. A right circular cone of diameter k and height 12 inches rests on the base of a right circular cylinder of radius k (their bases lie in the same plane, as shown). The cylinder is filled with water to a height of 12 inches. If the cone is then removed, compute the height to which the water will fall (that is, find the new depth of the water).

NOTE: For a cone, $V = \frac{1}{3}\pi r^2 h$. For a cylinder, $V = \pi r^2 h$

— SEE DIAGRAM AT BOTTOM OF PAGE —

T-4. From a set of 9 cards numbered consecutively from 1 to 9, two cards (numbered x and y) are chosen at random, with replacement. What is the probability that x is less than y?

T-5. ΔABC and $\Delta A'B'C'$ are equilateral triangles with parallel sides and the same center. The distance between side $\overline{BC}$ and side $\overline{B'C'}$ is one-sixth of the altitude of ΔABC. Find the ratio of the area of $\Delta A'B'C'$ to ΔABC.

— SEE DIAGRAM AT BOTTOM OF PAGE —

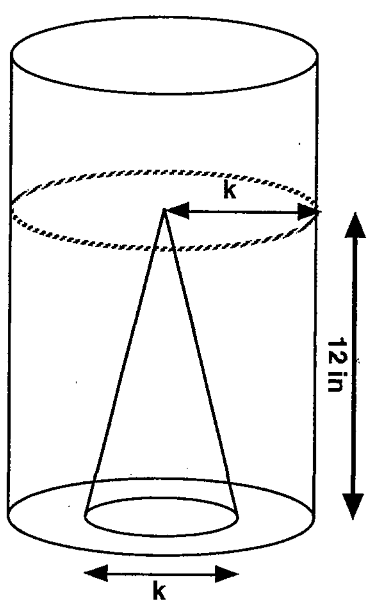


DIAGRAM FOR PROBLEM T–3

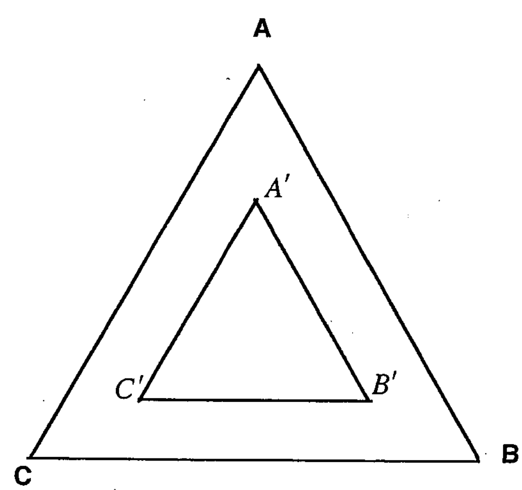


DIAGRAM FOR PROBLEM T–5

MCWP Senior High Mathematics League

Contest #3 **INDIVIDUAL QUESTIONS** **December 11, 2006**

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INDIVIDUAL ROUND 1 — 15 MINUTES

1. Three adjacent faces of a rectangular solid have respective areas of 6, 10, and 15. What is the sum of the lengths of ALL the edges of this solid?

2. When Red River flows downstream at a constant rate, I can cover the same distance rowing downstream for 10 minutes or upstream for 30. If I always row at the same rate, how many minutes would it take me to row this same distance if there were no current on Red River?

3. If x, y, and z are positive numbers, express $\dfrac{\dfrac{\frac{x}{y}}{z}}{\dfrac{z}{\frac{x}{y}}}$ in simplest form.

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INDIVIDUAL ROUND 2 — 15 MINUTES

4. In the coordinate plane, what is the length of the segment with endpoints $(a,0)$ and $(-b,0)$?

5. Pick an integer at random from the set of all 3-digit numbers $\{100, 101, 102, \ldots, 998, 999\}$. What is the probability that the integer you picked is a perfect square?

6. A *cevian* of a triangle is any segment whose endpoints are a vertex of the triangle and any non-vertex point on the opposite side. When I draw **1** cevian from each vertex of a triangle so that no interior point of the triangle is shared by three of the cevians, the interior of the triangle is split into **7** disjoint, non-overlapping regions. If the number 1 above is replaced with 2, then by what number should the 7 be replaced?

MCWP Senior High Mathematics League

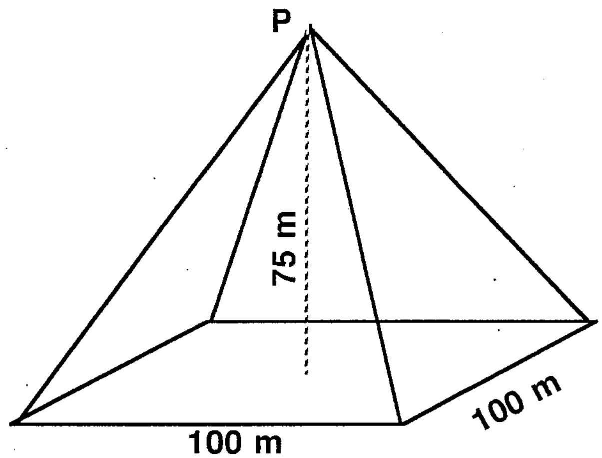
Contest #4 **January 8, 2007**

TEAM QUESTIONS

TIME LIMIT = 10 MINUTES

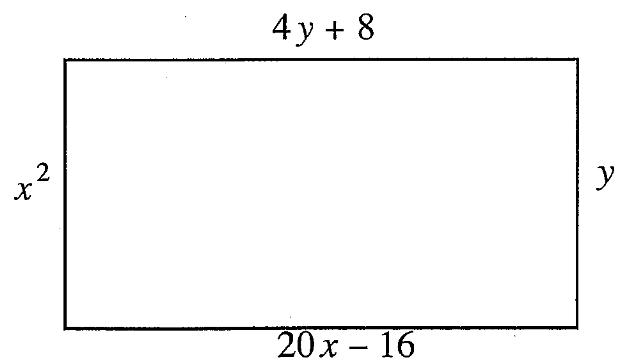
T-1. Mr. Reiter drives from home to work at an average speed of 40 miles per hour during rush hour. He leaves early and drives home at an average speed of 60 miles per hour. What is his average speed for the entire round trip from home to work and back?

T-2. A regular square pyramid has a square base with sides that are 100 meters long. Point P, the peak of the pyramid, is 75 meters above the base. What is the total surface area of the four triangular sides? (Leave your answer in simple radical form.)



T-3. What is the largest value of x less than 1 for which $\frac{2}{x}$ is an integer ?

T-4. A rectangle has the following sides, shown in the diagram. If at least one side must have an odd integer length, find the perimeter of the rectangle.



T-5. If $x^2 + x - 1 = 0$, what is the value of $x^4 + 2x^3 + x^2$?

MCWP Senior High Mathematics League

Contest #4 **INDIVIDUAL QUESTIONS** **January 8, 2007**

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INDIVIDUAL ROUND 1 — 15 MINUTES

1. What is the sum of the coefficients of x^2+ax+b, whose zeroes are $3+\sqrt{2}$ and $3-\sqrt{2}$?

2. What is the volume of a rectangular solid in which three adjacent faces have respective areas 24, 32, and 48?

3. Consider all pairs of integers (a,b), $0 < a < b < 100$, for which every integer greater than a and less than b is composite. For what pair (a,b) are these composites most numerous?

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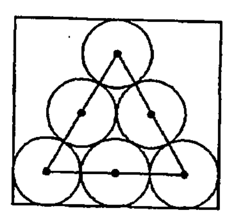
INDIVIDUAL ROUND 2 — 15 MINUTES

4. What are all values of x which satisfy $\left(\log_{10}x^2\right)^2 = \log_{10}10000$?

5. What value of x satisfies $5^x - 5^{x-2} = 120\sqrt{5}$?

6. In the diagram at the right, the area of each circle is π. If all the circles and segments which look tangent are tangent, what is the perimeter of the rectangle that surrounds these circles, as shown above?



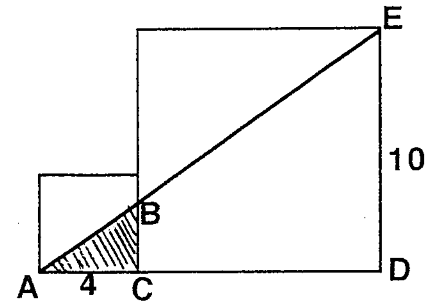
MCWP Senior High Mathematics League

Contest #5 **February 5, 2007**

TEAM QUESTIONS

TIME LIMIT = 10 MINUTES

T-1. Consider two squares with sides 4 and 10 as shown. What is the area of the shaded region?



T-2. A white urn contains ten white marbles and a black urn contains ten black marbles. The following events take place:

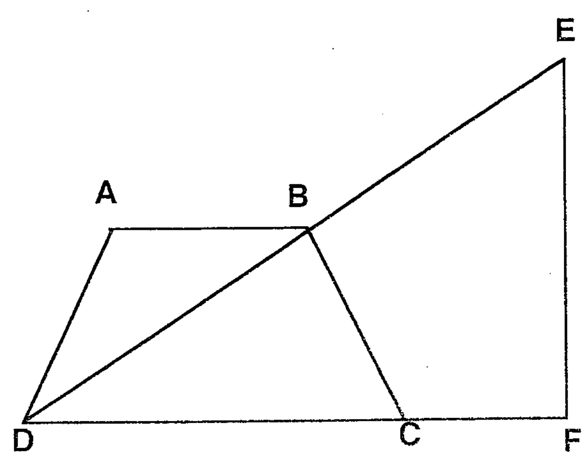
(a) First a marble is selected at random from the white urn and placed in the black urn.
(b) Then a marble is selected at random from the black urn and placed in the white urn.
(c) Finally, a marble is selected at random from the white urn.

What is the probability that the marble finally selected will be white?

T-3. For unequal integers x and y, the expression $3x + 8y$ takes on a range of different integral values. For example, if $x = 1$ *and* $y = -1$, the value of $3x + 8y$ *is* -5. In this manner, what is the smallest positive integral value of the expression, $3x + 8y$?

T-4. What is the smallest positive integer that leaves a remainder of 10 when divided into 200?

T-5. In the figure, $ABCD$ is an isosceles trapezoid with side lengths $AD = BC = 5$, $AB = 4$, *and* $DC = 10$. The point C is on the segment DF and B is the midpoint of the hypotenuse DE in the right triangle DEF. Find the length of segment CF.



MCWP Senior High Mathematics League

Contest #5 **INDIVIDUAL QUESTIONS** **February 5, 2007**

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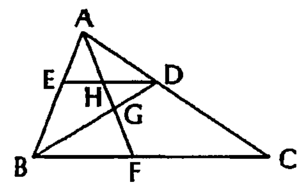
INDIVIDUAL ROUND 1 — 15 MINUTES

1. If the mean, median, and mode of the six numbers 6, 3, 5, 4, 5, and x are all equal to each other, what is the value of x?



2. How many different triangles can be traced along the lines in the diagram at the right?

3. What two-digit number greater than 10 is four times the sum and three times the product of its digits?

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INDIVIDUAL ROUND 2 — 15 MINUTES

4. What are all ordered pairs of numbers (x,y) which satisfy $x^2+y^2+x-y = 14$ and $xy+x-y = 1$?

5. What is the area (in m^2) of the largest flat parcel of land that can be surrounded by the same rope that can just surround a flat, square parcel of land with an area of 3600 m^2?

6. What are both values of a for which the line connecting $P(a-1,a-3)$ to $(2,-3)$ is perpendicular to the line connecting P to $(-1,2)$?

MCWP Senior High Mathematics League

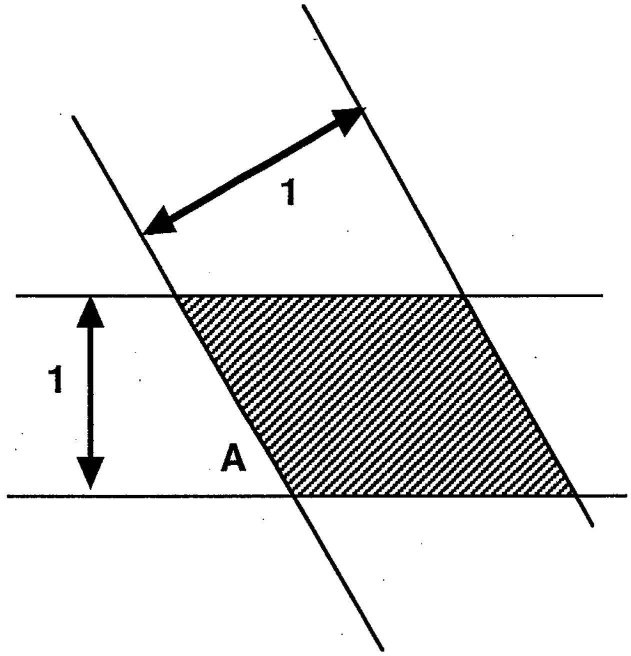
Contest #6 **March 5, 2007**

TEAM QUESTIONS

TIME LIMIT = 10 MINUTES

T-1. If you pick a positive factor of 144 at random, what is the probability that it is a perfect square?

T-2. Two strips of width 1-inch overlap at an angle A as shown. What is the area of the overlap (shown shaded). Your answer should be in stated in terms of angle A?



T-3. If the length of each side of a triangle is increased by 20%, then the area of the triangle is increased by what percent?

T-4. In a certain circle, the numerical sum of the area and circumference is 360π. How long is the radius of this circle?

T-5. The set $S = \left\{\frac{1}{15}, \frac{2}{15}, \frac{4}{15}, \frac{7}{15}, \frac{8}{15}, \cdots \frac{74}{15}\right\}$ consists of every positive fraction whose denominator is 15, whose numerator is at most 74, and whose numerator and denominator are integers with a greatest common factor of 1. What is the sum of all the elements of S ?

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INDIVIDUAL ROUND 1 — 15 MINUTES

1. Factor $12x^2 - 13x - 35$ into a product of linear binomials with integer coefficients.

2. What real value of x satisfies $\log_{16}x^2 + \log_4(-x) = 3$?

3. The total number of sides of two regular polygons is 10 and their total number of diagonals is 11. What is the sum of the degree-measures of all their interior angles?

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INDIVIDUAL ROUND 2 — 15 MINUTES

4. What are all real numbers x which satisfy $(x^2-3x-3)^2 > (x^2+7x-13)^2$?

5. I have 87 vims, of which 34 are mims and 49 are pims. If exactly 9 vims are both mims and pims, then how many vims are neither pims nor mims?

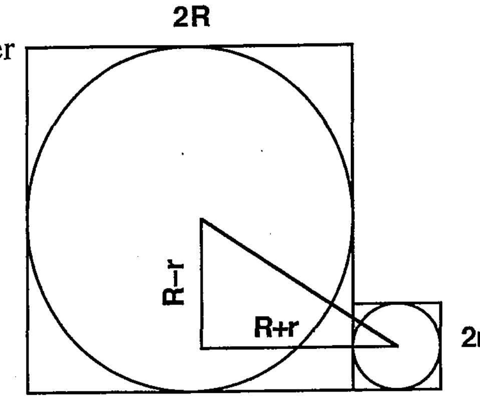
6. If $0° \le \theta < 90°$, what are all values of θ which satisfy $\tan(3\theta - 10°) = \cot(\theta + 60°)$?

SOLUTIONS TO TEAM QUESTIONS

Contest #1, 2006 - 2007 **October 16, 2006**

T-1. The sum of the interior angles of a polygon of n sides is $180(n-2)$. Then we have, $133(n-1)+x=180(n-2)$ from which $x=47n-227$. Since x is between 0 and 180, we can only have $n=4,\ 6,\ 7,\ 8$, producing $x=8,\ 55,\ 102,\ 149$

T-2. Let the side of the larger square be 2R and the side of the smaller square be 2r, where R and r are also the radii of the 2 circles.
Then $(2R)^2=49(2r)^2$ so that $R^2=49r^2$.
Also, $(2R)^2+(2r)^2=200$ so that $R^2+r^2=50$.
Substituting, we have $49(r)^2+r^2=50$.
So, $r=1$ *and* $R=7$. The distance between the centers is given by the Pythagorean Theorem. (see diagram).



$$\sqrt{(R+r)^2+(R-r)^2}=\sqrt{2R^2+2r^2}=\sqrt{2(7)^2+2(1)^2}=10$$

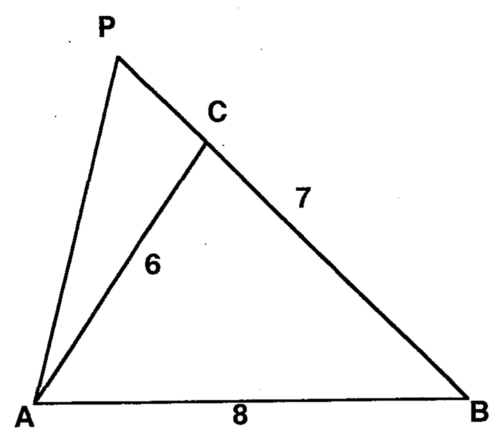
T-3. The area of the entire 12-inch target is 144π. Consider 2 circles which have radii 2 inches and 6 inches. The area of the larger circle is 36π, and the area of the smaller one is 4π. The area of the band between these two circles is 32π. Therefore, the probability that the dart will land in this band is $\frac{32\pi}{144\pi}=\frac{2}{9}$

T-4. From the similar triangles $\Delta PAB \approx \Delta PCA$, we have

$$\frac{PA}{PC}=\frac{AB}{CA}=\frac{PB}{PA} \quad or \quad \frac{PA}{x}=\frac{8}{6}=\frac{PB}{PA}$$

So, then $PA=\frac{4}{3}x=\frac{3}{4}PB$ *so* $16x=9PB$

Since $PB=7+x$, *we have* $16x=9(7+x)$, So $x=9$.



T-5. Let x be the radius of a tennis ball. Then the total volume of the 3 tennis balls is

$V=3\left(\frac{4}{3}\pi x^3\right)=4\pi x^3$. The volume of the can is $V=\pi x^2 h=\pi x^2(6x)=6\pi x^3$.

The volume of the unused space inside the can is the difference of these two, or $2\pi x^3$

The required ratio is $\frac{4\pi x^3}{2\pi x^3}=\frac{2}{1}$ *or* 2:1

SOLUTIONS TO INDIVIDUAL QUESTIONS

Contest #1, 2006-2007 **October 16, 2006**

I–1. For the line to pass through all three points, the slope calculated using any pair of points must be the same. Therefore we have,
$$\frac{3-k}{1-(-2)} = \frac{3-(-4)}{1-7}.$$
Simplifying, we have $\frac{3-k}{3} = \frac{7}{-6}$ so, $k = \frac{13}{2}$

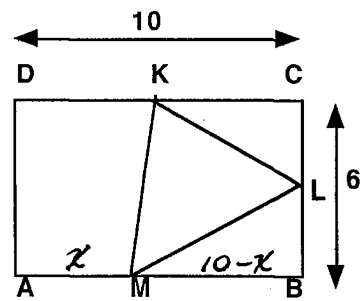
I–2. What we need to calculate is the probability of getting two \$10 bills, or two \$5 bills, or two \$1 bills on the first two draws. The probability of getting two \$10's is $\left(\frac{4}{12}\right)\left(\frac{3}{11}\right) = \frac{12}{(12)(11)} = \frac{1}{11}$.

The probability of getting two \$5's is $\left(\frac{6}{12}\right)\left(\frac{5}{11}\right) = \frac{30}{(12)(11)} = \frac{5}{22}$

The probability of getting two \$1's is $\left(\frac{2}{12}\right)\left(\frac{1}{11}\right) = \frac{2}{(12)(11)} = \frac{1}{66}$

So the probability that one of the above three events will occur is the sum of these, or $\frac{6+15+1}{66} = \frac{22}{66} = \frac{1}{3}$

I–3. Let $x = AM$, the required length.
Then $MB = 10 - x$
The rectangle is divided into 3 triangles and a trapezoid.
Adding these 4 areas, we have



$$60 = \frac{1}{2}(5)(3) + \frac{1}{2}(3)(10-x) + 12 + \frac{1}{2}(6)(5+x)$$

This reduces to $\frac{15}{2} + 15 - \frac{3}{2}x + 12 + 15 + 3x = 60$

so that $49\frac{1}{2} + \frac{3}{2}x = 60$ from which we find that

$99 + 3x = 120$ *and* $x = 7$

I–4. Since the sum of the angles of a trapezoie is 360°, we have $(A) + (A+d) + (A+2d) + (A+3d) = 360$, which simplifies to $4A + 6d = 360$. Since A = 75°, we then have $300 + 6d = 360$ and $d = 10$
So, the largest angle is $A + 3d = 75 + 30 = 105°$

I–5. Since x and y are inversely proportional, we have $y = k\left(\frac{1}{x}\right)$.

When x is increased by p%, let a% be the corresponding percent decrease in y.

Then we have, $y\left(1 - \frac{a}{100}\right) = k\left(\frac{1}{x\left(1+\frac{p}{100}\right)}\right) = \frac{k}{x}\left(\frac{100}{100+p}\right)$.

But since $y = \frac{k}{x}$, we have $y\left(1 - \frac{a}{100}\right) = y\left(\frac{100}{100+p}\right)$.

Solving for a, we have $a = \frac{100p}{100+p}$

I–6. Initially, the female membership is 1%, or 5 members. Therefore, when 5 females account for 4% of the club, the total membership is $\frac{5}{.04} = 125$

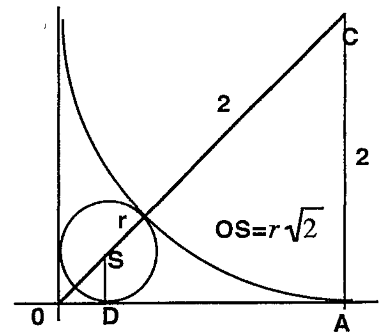
SOLUTIONS TO TEAM QUESTIONS

Contest #2, 2006 - 2007 **November 13, 2006**

T-1. Let p be the number of students who passed, and f be the number of students who failed. Then the total number of points earned by all students is given by $65p + 35f = 53(p+f)$. So we then have $2p = 3f \Rightarrow f = \frac{2}{3}p$.

The total number of students is $f + p = \frac{2}{3}p + p = \frac{5}{3}p$. Then the percentage of the students who passed is $\dfrac{p}{\left(\frac{5}{3}p\right)} = \frac{3}{5} = 60\%$. (The information that passing is 60 points or better, is not needed for the solution.)

T-2. Draw the line $\overline{OC}$, which passes through S. Also draw $\overline{SD}$ *and* $\overline{CA}$ perpendicular to $\overline{OA}$. Then ΔOSD *and* ΔOCA are similar triangles. Let r be the radius of circle S. We see that $\angle SOD$ is 45°, so that $SO = r\sqrt{2}$. Since the radius of circle C is 2, we have: $\dfrac{2 + r + r\sqrt{2}}{r\sqrt{2}} = \dfrac{2}{r}$. Solving for r, we have $r = \dfrac{2(\sqrt{2}-1)}{\sqrt{2}+1} = 6 - 4\sqrt{2}$

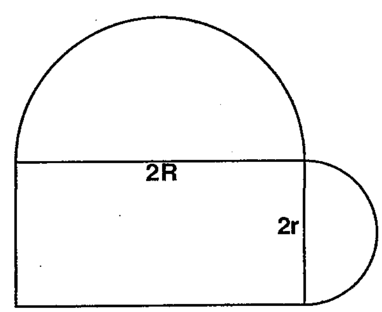


T-3. Since $(1 + x + y)^2 = 1 + x^2 + y^2 + 2x + 2y + 2xy = 1 + x^2 + y^2$ simplifies to $x + y + xy = 0$, we can use this simpler equation to solve for y, getting $y = -\dfrac{x}{x+1} = -\dfrac{x+1-1}{x+1} = -1 + \dfrac{1}{x+1}$. Since x and y are integers, the denominator, $x+1$, must be a divisor of the numerator, 1.
The possibilities are $x + 1 = 1$ *or* $x + 1 = -1$. Thus, $x = 0$ *or* $x = -2$, and (x,y) *is either* $(0,0)$ *or* $(-2,-2)$.

T-4. Let R and r be the radii of the circles. Then the area of the rectangle is $(2R)(2r) = 81$. Since the areas of the circles are in the ratio of 16:1, we have

$$\frac{\left(\frac{1}{2}\right)\pi R^2}{\left(\frac{1}{2}\right)\pi r^2} = \frac{16}{1} \text{ so that } R = 4r.$$

Therefore, $(2R)(2r) = (8r)(2r) = 81$, *so* $r = \frac{9}{4}$ *and* $R = 9$.
Then the perimeter of the rectangle is $2(2R) + 2(2r) = 36 + 9 = 45$



T-5. If we rewrite the inequality as

$$\frac{2006-1}{2006} < \frac{a}{b} < \frac{2007-1}{2007} \quad or \quad 1 - \frac{1}{2006} < \frac{a}{b} < 1 - \frac{1}{2007},$$

we then can write $-\dfrac{1}{2006} < \dfrac{a}{b} - 1 < -\dfrac{1}{2007}$ or $-\dfrac{1}{2006} < \dfrac{a-b}{b} < -\dfrac{1}{2007}$

Then, multiplying by (–1), we have $\dfrac{1}{2007} < \dfrac{b-a}{b} < \dfrac{1}{2006}$

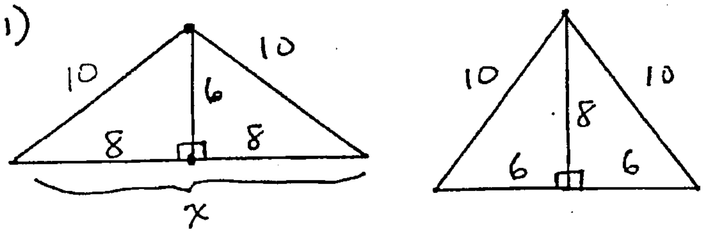
Therefore, $2006(b-a) < b < 2007(b-a)$.
The least possible integer, b, occurs when $b - a = 2$ and b = 4013

Math Council of Western PA Sr High Math Leauge 2006-2007
SOLUTIONS TO INDIVIDUAL QUESTIONS
Contest #2 — November 13, 2006

Answers

1. 16
2. $\frac{5}{11}$
3. 7
4. $\{x | x > -1/2,\ x \neq 1\}$ or $\{x | -1/2 < x < 1 \text{ or } x > 1\}$ or $(-1/2, 1) \cup (1, \infty)$
5. 44/15
6. -1

1)



other

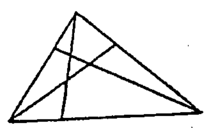
$x = 8+8 = \boxed{16}$.

2)
1 dime, 1 quarter : $\frac{1}{3}\left(\frac{1}{2}\right) = \frac{1}{6} = \frac{3}{18}$ $P = \frac{3}{3+2+6} = \frac{3}{11}$

1 dime, 2 quarters : $\frac{1}{3}\left(\frac{1}{3}\right) = \frac{1}{9} = \frac{2}{18}$ $P = \frac{2}{11}$ $\Big\}\ \boxed{\frac{5}{11}}$

2 dimes : $\frac{1}{3}(1) = \frac{1}{3} = \frac{6}{18}$ $P = \frac{6}{11}$ $\Big\}\ \frac{6}{11}$

3)



$\boxed{7}$

4) Clearly, $x \neq 1$. Then $x+3 > -x+2$, or $x > -\frac{1}{2}$

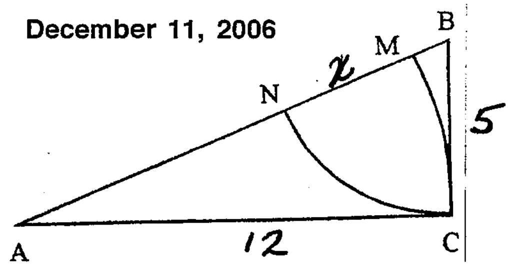
$\therefore \boxed{\{x | x > -\frac{1}{2} \text{ and } x \neq 1\}}$

5) $\frac{5+16+21+16+30}{5+8+7+4+6} = \frac{88}{30} = \boxed{\frac{44}{15}}$.

6) This is the binomial expansion of $(86-87)^5 = (-1)^5 = \boxed{-1}$.

SOLUTIONS TO TEAM QUESTIONS

Contest #3, 2006 - 2007 **December 11, 2006**



T-1. This is a 5–12–13 right triangle. Let $x = MN$.
Since the arc CM is drawn with center at A, $AM = AC = 12$.
Since the arc NC is drawn with center at B, $BN = BC = 5$.
Also, since $AB = 13$, $BM = 13 - AM = 13 - 12 = 1$.
Then it follows that $MN = BN - BM = 5 - 1 = 4$.

T-2. **METHOD I**

Since $\frac{1}{a} + \frac{1}{b} = \frac{a+b}{ab}$, we need to evaluate $\frac{a+b}{ab}$.

Consider $\frac{1}{a(b+1)} + \frac{1}{b(a+1)} = \frac{1}{(a+b)(b+1)}$ and clear of fractions to get: $b(a+1) + a(b+1) = ab$.

Expanding the left side and simplifying, we get $a + b = -ab$. Then, dividing both sides by ab we get $\frac{a+b}{ab} = -1$

METHOD II

Multiply by $(a+1)(b+1)$ produces $\frac{a+1}{a} + \frac{b+1}{b} = 1 + \frac{1}{a} + 1 + \frac{1}{b} = 1$; therefore, $\frac{1}{a} + \frac{1}{b} = -1$

T3. The volume of a cylinder is $V = \pi r^2 h$ and the volume of a cone is $V = \frac{1}{3}\pi r^2 h$.

The amount of water displaced by the cone is $V = \frac{1}{3}\pi\left(\frac{k}{2}\right)^2(12)$.

Therefore, the final volume of the water is $\pi k^2(x) = \pi k^2(12) - \frac{1}{3}\pi\left(\frac{k}{2}\right)^2 12 = \pi k^2(11)$. So, x=11.

T-4. **METHOD I**

The probability that $x = y$ is $\frac{1}{9}$, so the probability that $x \neq y$ is $\frac{8}{9}$. Since x and y are chosen at random, it's equally likely that $x < y$ *or* $x > y$; so the probability that $x < y$ equals $\left(\frac{1}{2}\right)\left(\frac{8}{9}\right) = \frac{4}{9}$

METHOD II

If we just list the pairs, there are $9^2 = 81$ ordered pairs in all. The ones for which $x < y$ are:

$(1,2),\ (1,3),\ (1,4),\ \cdots\ (1,9)$ — 8 *pairs*

$(2,3),\ (2,4),\ (2,5)\cdots\ (1,9)$ — 7 *pairs*

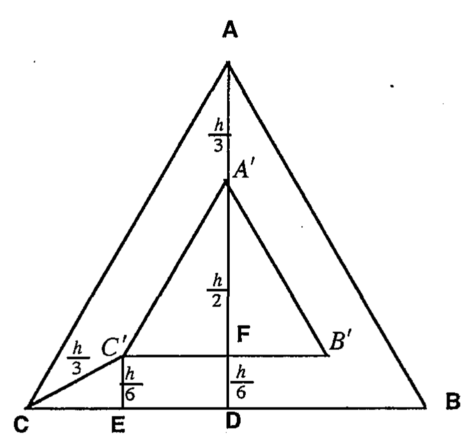
$(8,9)$ — 1 *pair*

Since $1 + 2 + 3 + \cdots + 7 + 8 = 36$, the required probability is $\frac{36}{81} = \frac{4}{9}$

T-5. Draw $AD = h$, the altitude of ΔABC. It is given that $DF = EC' = \frac{1}{6}h$. $\Delta CC'E$ is a 30°–60°–90° right triangle, so $CC' = AA' = \frac{2h}{6} = \frac{h}{3}$. Subtracting, we find that $A'F = \frac{3h}{6} = \frac{h}{2}$. Then by similar triangles, $\frac{A'F}{AD} = \frac{C'B'}{CB} \Rightarrow \frac{\frac{h}{2}}{h} = \frac{C'B'}{CB}$.

Then, $C'B' = \frac{CB}{2}$, so that the ratio of the areas is:

$$\frac{\frac{1}{2}(C'B')(A'F)}{\frac{1}{2}(CB)(h)} = \frac{\frac{1}{2}\left(\frac{1}{2}CB\right)\left(\frac{1}{2}h\right)}{\frac{1}{2}(CB)(h)} = \frac{1}{4} \text{ or } 1:4.$$

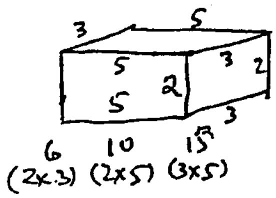


Math Council of Western PA Sr High Math Leauge 2006 - 2007
SOLUTIONS TO INDIVIDUAL QUESTIONS
Contest #3 — December 11, 2006

Answers

1. 40
2. 15 or 15 mins.
3. x^2
4. $|a+b|$ or $|-a-b|$ or $\sqrt{(a+b)^2}$ or $\sqrt{(-a-b)^2}$
5. 22/900 or 11/450 or $0.02\overline{4}$
6. 19

1)



Each edge-length is repeated 3 additional times. Sum $= 4(2+3+5) = \boxed{40}$.

2)

Rate	ΔTime	Distance
$r+c$	10 mins.	1 unit
$r-c$	30 mins.	1 unit

$10r + 10c = 30r - 30c$

$r = 2c$

$r+c = 3c$

$r-c = c$

R	$\Delta T_{mins.}$	D
$3c$	10	$30c$
$r = 2c$	$\boxed{15}$	$30c$
c	30	$30c$

3)

$$\frac{x}{\frac{y}{z}} = \frac{xz}{y}, \qquad \frac{\frac{z}{x}}{y} = \frac{z}{xy}$$

$$\frac{xz}{y} \div \frac{z}{xy} = \left(\frac{xz}{y}\right)\left(\frac{xy}{z}\right) = \boxed{x^2}.$$

4) By the distance formula, the length is $\boxed{\sqrt{(a+b)^2}}$. Since the points lie on the x-axis, $\boxed{|a+b|}$, or its equivalents, is also correct

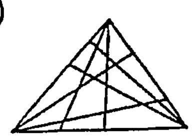
5)

$$22\left\{\begin{array}{l} 10^2 = 100 \\ \\ 31^2 = 961 \end{array}\right.$$

$32^2 = 1024$

$$P = \frac{22}{900} = \boxed{\frac{11}{450}}.$$

6)



$\boxed{19}$

<u>SOLUTIONS TO TEAM QUESTIONS</u>

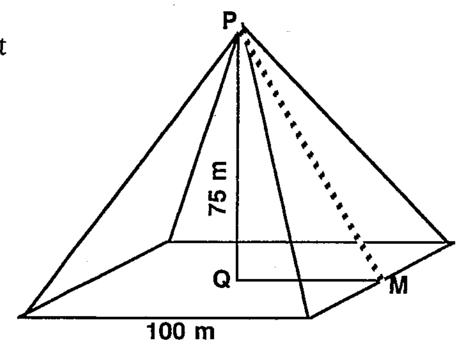
Contest #4, 2006 - 2007 **January 8, 2007**

T-1. The average speed for the round trip is found by finding total distance divided by total time for the round trip. Let d be the one-way distance. The time from home to work is $d/40$ and the time from work to home is $d/60$.

So total distance over total time is: $\dfrac{2d}{\frac{d}{40}+\frac{d}{60}} = \dfrac{2d}{\frac{100d}{2400}} = \dfrac{4800d}{100d} = 48\ mph$

T-2. Let PQ be the altitude of the pyramid, which is 75m. Draw QM to the midpoint of one side of the base. QM = 50. Then the height, PM, of one of the triangular sides is $\sqrt{75^2+50^2} = \sqrt{25^2(9+4)} = 25\sqrt{13}$.

So, the area of the four triangular sides is $4\left(\frac{1}{2}\right)(100)\left(25\sqrt{13}\right) = 5000\sqrt{13}$.



T-3. **METHOD I**

We want the largest value of x, so try to find a positive value of x, not a negative one. For $0 < x < 1$, we know that $\frac{2}{x} \geq 2$. When $\frac{2}{x}$ attains its least integral value, x will attain its greatest possible value. The least integer greater than 2 is 3, and when $\frac{2}{x} = 3$, the value of x is $\frac{2}{3}$

METHOD II

Let $\frac{2}{x} = n$, so $x = \frac{2}{n}$. As we did in METHOD I, above, let's require that $0 < x < 1$; so $n > 2$. The least possible integral value of n is 3. This value of n yields the greatest value of x, which is $\frac{2}{3}$.

T-4. Since the opposite sides of the rectangle are equal, we have $4y + 8 = 20x - 16$ and $y = x^2$.
Solving this system of equations by substituting the second equation into the first, we have $4x^2 + 8 = 20x - 16$.
Simplifying, $4x^2 - 20x + 24 = 0$ *and* $x^2 - 5x + 6 = 0$, so $x = 2,\ 3$.
Therefore, either $x = 2$ *and* $y = 4$, *or* $x = 3$ *and* $y = 9$. Since at least one side must have an odd integer length, the first set of answers is discarded. Then the perimeter is: $2(9 + 44) = 106$.

T-5. **METHOD I**

$\left(x^2 + x\right)^2 = 1^2 \Rightarrow x^4 + 2x^3 + x^2 = 1$

METHOD II

Since $x^2 + x - 1 = 0$, we have $x^2 = 1 - x$.

Therefore, $x^4 + 2x^3 + x^2 = \left(x^2\right)^2 + 2\left(x^2\right)x + x^2$

$= (1-x)^2 + 2(1-x)x + x^2 = 1 - 2x + x^2 + 2x - 2x^2 + x^2 = 1$

Math Council of Western PA Sr High Math Leauge 2006 - 2007
SOLUTIONS TO INDIVIDUAL QUESTIONS
Contest #4 — January 8, 2007

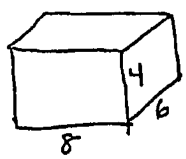
Answers

1. 2	2. 192	3. (89,97)
4. $\pm 10, \pm\frac{1}{10}$	5. 3.5	6. $16+4\sqrt{3}$

1)

$$(x-(3-\sqrt{2}))(x-(3+\sqrt{2}))$$
$$= x^2-3x+\cancel{x\sqrt{2}}-3x-\cancel{x\sqrt{2}}+(9-2)$$
$$= x^2-6x+7.$$ Sum of coefficients is $1-6+7=\boxed{2}$

2)



$24 = 3\times 8$ or $\underline{4\times 6}$ or 2×12

$32 = 2\times 16$ or $\underline{4\times 8}$

$48 = 3\times 16$ or 4×12 or $\underline{6\times 8}$

$4\times 6\times 8 = \boxed{192}$.

3)

89, 90, 91, 92, 93, 94, 95, 96, 97

89: prime; 97: prime $\boxed{(89, 97)}$

divisible by 2 7 2 3 2 5 2

No other pair of consecutive primes < 100 is this far apart.

4)

$(\log_{10}x^2)^2 = \log_{10}10\,000 = 4$, so $\log_{10}x^2 = \pm 2$

$x^2 = 100$ or $\frac{1}{100}$

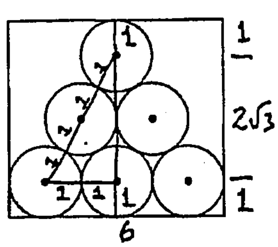
$x = \boxed{\pm 10, \pm\frac{1}{10}}$

5)

$5^x - \frac{5^x}{25} = 120\sqrt{5} \Leftrightarrow \frac{\cancel{24}\cdot 5^x}{25} = \cancel{120}^{\,5}\sqrt{5}$, so $5^x = 125\sqrt{5} = 5^{3.5}$,

so $x = \boxed{3.5}$

6)



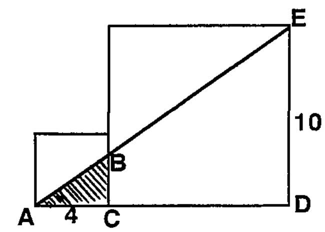
$P = 2(2+2\sqrt{3})+2(6) = \boxed{16+4\sqrt{3}}$.

SOLUTIONS TO TEAM QUESTIONS

Contest #5, 2006 - 2007 **February 5, 2007**

T-1. Triangle ACB and ADE are similar. Therefore, $\frac{BC}{4} = \frac{10}{14} \Rightarrow BC = \frac{20}{7}$

So, the area of ΔABC *is* $\left(\frac{1}{2}\right)(4)\left(\frac{20}{7}\right) = \frac{40}{7}$



T-2. After event (a) is accomplished, the black urn contains 10 black marbles and 1 white marble. After event (b) is accomplished, there are two possible outcomes —

(1) The white urn could contain 10 white marbles -- probability = $\frac{1}{11}$.

In this case, the probability of choosing a white marble on the next draw is 1.

(2) The white urn could contain 9 white marbles and 1 black marble -- probability=$\frac{10}{11}$

In this case, the probability of choosing a white marble on the next draw is $\frac{9}{10}$

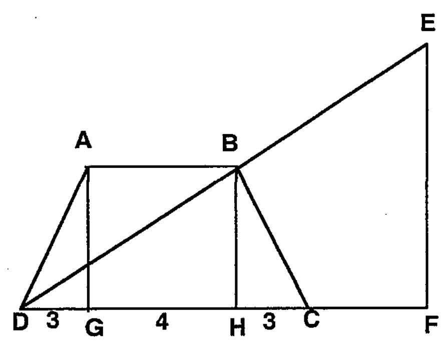
Adding the two results, the probability that the final ball chosen in white is $\left(\frac{1}{11}\right)(1) + \left(\frac{10}{11}\right)\left(\frac{9}{10}\right) = \frac{10}{11}$

T-3. By trial and error, if $x = 3$, and $y = -1$, the value of $3x + 8y$ is 1.

NOTE: It is a theorem that, if the greatest common divisor of a and b is 1, then there always exist integers x and y for which $ax + by = 1$. Here, $a = 3$ and $b = 8$.

T-4. Any number that leaves a remainder of 10 when divided into 200 must be a factor of $200 - 10 = 190$. The prime factors of 190 are 2, 5, and 19. The smallest factor of 190 that's *greater than* 10 is 19

T-5. Draw AG *and* BH perpendicular to $\overline{DF}$.
Then we see that $DG = 3$, $GH = 4$, *and* $HC = 3$.
Now B is the midpoint of DE. Using similar triangles,
we see that the sides of ΔDEF are all twice as big as the sides of ΔDBH.
Therefore, $DH = HF = 7$, *and* $CF = 4$.



Math Council of Western PA Sr High Math Leauge 2006 - 2007
SOLUTIONS TO INDIVIDUAL QUESTIONS
Contest #5 — February 5, 2007

Answers

1. 7 2. 13 (*ABC, ABD, ABF, ABG, ACF, ADE, ADG, ADH, BCD, BDE, BFG,* DGH, AEH)

3. 24 4. (−4,−1), (1,−3), (1,4), (3,−1) 5. $\frac{14400}{\pi}$ or $\frac{14400}{\pi}$ m^2 or 4583.66236 . . . 6. 0, 4

1) The mode of 6, 3, 5, 4, 5, and x must be 5, since the problem statement implies there is only one mode. Since the mean is also 5, $\frac{6+3+5+4+5+x}{6}=5$, Solving, $x=\boxed{7}$ – and in this case, median is also 5.

2) $\boxed{13}$ (*ABC, ABD, ABF, ABG, ACF, ADE, ADG, ADH, BCD, BDE, BFG,* DGH, AEH)

3) $10t+u = 4(t+u) \Rightarrow u = 2t$

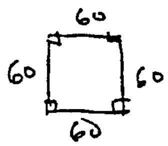
$10t+u = 3tu$, so $12t = 6t^2$, Since $t \neq 0$, $t=2$, $u=4$, # is $\boxed{24}$.

4) From second equation, $x(y+1) = (y+1)$.

Thus, either $x=1$ or $y=-1$. Direct substitution produces the solutions $\boxed{(-4,-1), (1,-3), (1,4), (3,-1)}$.

or $x^2+y^2+x-y-2(xy+x-y) = 14-2 \Rightarrow (x-y)^2-(x-y)-12=0$, so $x-y=4$ or $x-y=-3$ etc.

5)



240 m of rope. Form a circle. $240 = 2\pi r$, so $r = \frac{120}{\pi}$ m

and $\pi\left(\frac{120}{\pi}\right)^2 = \boxed{\frac{14400}{\pi}\text{ m}^2}$.

6) $P(a-1, a-3)$ $m_{AP} = \frac{a}{a-3}$ Case I: $m_1 m_2 = -1 \Rightarrow \frac{\cancel{a}}{a-3}\cdot\frac{a-5}{\cancel{a}} = -1$; $\boxed{a=4}$

$A(2,-3)$ $m_{PB} = \frac{a-5}{a}$ Case II: $m_1 = 0$, $m_2 =$ undefined $\boxed{a=0}$

$B(-1,2)$ Case III $m_1 =$ undef, $m_2 = 0$ NO

SOLUTIONS TO TEAM QUESTIONS

Contest #6, 2006 - 2007 **March 5, 2007**

T-1. The prime factorization of 144 is: $2^4 3^2$. So the factors of 144 are:
1, 2, 3, 4, 6, 8, 9, 12, 16, 18, 48, 72, 24, 36, 144.
Then the probability that the factor chosen is a perfect square is $\frac{6}{15} = \frac{2}{5}$

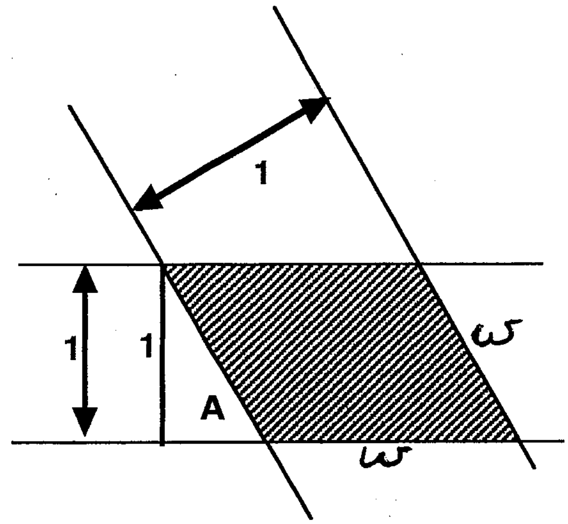
T-2. Let w be the length of one side of the rhombus.

The shaded area is a rhombus with area,

$Area = (base)(height) = w(1) = w$.

But, $\sin A = \dfrac{1}{w}$

Therefore, $Area = w = \dfrac{1}{\sin A}$.



T-3. With each side increased by 20%, the area $= \frac{1}{2}(1.2b)(1.2h) = .72bh$.

Using b and h, the area $= \frac{1}{2}bh = .5bh$

The increase in area is $.22bh$. The percent increase is $\dfrac{.22bh}{.5bh} = .44$ *or* 44%

T-4. $A = \pi r^2$ *and* $C = 2\pi r$, so $\pi r^2 + 2\pi r = 360\pi \Rightarrow r^2 + 2r = 360$.
Since $r > 0$, $r^2 + 2r - 360 = (r - 18)(r + 20) = 0 \Rightarrow r = 18$

T-5. $S = \left\{\frac{1}{15}, \frac{2}{15}, \frac{4}{15}, \frac{7}{15}, \frac{8}{15}, \cdots \frac{74}{15}\right\}$. The numerators of the first 8 terms are the whole numbers less than 15 that are divisible by neither 3 nor 5. The sum of these 8 fractions is 4, easily obtained by the pairings, $\left(\frac{1}{15} + \frac{14}{15}\right) + \left(\frac{2}{15} + \frac{13}{15}\right) + \left(\frac{3}{15} + \frac{12}{15}\right) + \left(\frac{4}{15} + \frac{11}{15}\right) = 4$. To get the next 8 terms, just add 1 to each of the first 8. We then get $1\frac{1}{15}, 1\frac{2}{15}, 1\frac{4}{15} \cdots 1\frac{14}{15}$. The sum of these 8 terms is $8 \times 1 + 4$. Continuing in a similar fashion, we find that the sum of all terms in S is
$(8 \times 0 + 4) + (8 \times 1 + 4) + (8 \times 2 + 4) + (8 \times 3 + 4) + (8 \times 4 + 4)$
$= 8(0 + 1 + 2 + 3 + 4) + 5(4) = 80 + 20 = 100$

Math Council of Western PA Sr High Math Leauge 2006 - 2007
SOLUTIONS TO INDIVIDUAL QUESTIONS
Contest #6 — March 5, 2007

Answers

1. $(3x-7)(4x+5)$
2. -8
3. 1080 or 1080°
4. $\{x \mid x<-4 \text{ or } 1<x<2\}$ or equivalent disjunction
5. 13
6. 10°, 55° (both reqd)

1) $12x^2 \underline{-13x} - 35 = 12x^2 \underline{-28x + 15x} - 35 = 4x(3x-7) + 5(3x-7)$
$= (3x-7)(4x+5)$

2) $\log_{16} x^2 + \underbrace{\log_4(-x)}_{\therefore\, x<0} = 3$. Since $\log_{b^2} a^2 = \log_{|b|}|a|$, $\log_{16} x^2 = \log_4(-x)$

$2\log_4(-x) = 3 \Leftrightarrow 4^{3/2} = -x$, so $x = \boxed{-8}$.

3)

# sides	3	4	5	6
# diags	0	2	5	9

quad + hex $= 360° + 720° = \boxed{1080°}$.

4) $(x^2-3x-3)^2 - (x^2+7x-13)^2 > 0$

$(x^2-3x-3-x^2-7x+13)(x^2-3x-3+x^2+7x-13) > 0$

$(-10x+10)(2x^2+4x-16) > 0 \Leftrightarrow (x-1)(x-2)(x+4) < 0$

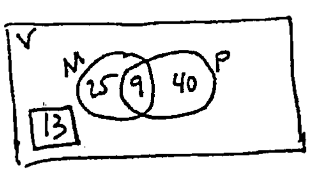
$(-10)(x-1)(2)(x-2)(x+4) > 0$

product $-\ 0\ +\ 0\ -\ 0\ +$

x: -4, 1, 2

$\boxed{\{x \mid x<-4 \text{ or } 1<x<2\}}$

5)



6) $\tan(3\theta - 10°) = \cot(\theta + 60°)$

$(3\theta - 10°) + (\theta + 60°) = 90° + 180k°$

$4\theta = 40° + 180k°$

$\theta = 10° + 45k°$ Acute ∡ solutions are $\boxed{10°, 55°}$